

Controlling ρ width effects for a precise value of α in $B \rightarrow \rho\rho$

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It has been pointed out that the currently most precise determination of the weak phase $\phi_2 = \alpha$ of the Cabibbo-Kobayashi-Maskawa (CKM) matrix achieved in $B \rightarrow \rho\rho$ decays is susceptible to a small correction at a level of $(\Gamma_\rho/m_\rho)^2$ due to an $I = 1$ amplitude caused by the ρ width. Using Breit-Wigner distributions for the two pairs of pions forming ρ mesons, we study the $I = 1$ contribution to $B \rightarrow \rho\rho$ decay rates as function of the width and location of the ρ band. We find that in the absence of a particular enhancement of the $I = 1$ amplitude reducing a single band to a width Γ_ρ at SuperKEKB leads to results which are completely insensitive to the ρ width. If the $I = 1$ amplitude is dynamically enhanced relative to the $I = 0, 2$ amplitude one could subject its contribution to a “magnifying glass” measurement using two separated ρ bands of width Γ_ρ . Subtraction of the $I = 1$ contribution from the measured decay rate would lead to a very precise determination of the $I = 0, 2$ amplitude needed for performing the isospin analysis.

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1 Introduction

Precision measurements of phases in the Cabibbo-Kobayashi-Maskawa (CKM) matrix responsible for CP violation are one way for detecting new physics. The most accurate method for determining $\phi_2 = \alpha \equiv \text{Arg}(-V_{tb}^* V_{td}/V_{ub}^* V_{ud})$ is based on $B \rightarrow \rho\rho$ decays. Isospin symmetry implies amplitude triangle relations for longitudinally polarized ρ mesons in $B^0 \rightarrow \rho^+ \rho^-, \rho^0 \rho^0$ and $B^+ \rightarrow \rho^+ \rho^0$ and their charge-conjugates, which specify a value for α when being augmented by time-dependent CP asymmetries in the first two processes [1]. The current status of applying this method has been described recently in Ref. [2], leading to a present error of 5° in α , and projecting an error less than one degree for future experiments to be performed by the Belle II Collaboration at SuperKEKB [3, 4].

In the above method one neglects the ρ width assuming equal masses for the two final ρ mesons, which by Bose symmetry must be in $I = 0$ and $I = 2$ states. The authors of Ref. [5] pointed out that the ρ width introduces a new isospin amplitude, because two ρ mesons observed with different invariant masses may have a total $I = 1$. The $I = 1$ contribution to

$B \rightarrow \rho\rho$ decay rates was suggested on dimensional grounds to decrease with the width Δ of the ρ band at least as $(\Delta/m_\rho)^2$ for $\Delta < \Gamma_\rho$. In order to eliminate corrections in α due to the ρ width Ref. [5] proposed to measure decay rates for decreasing values of Δ , reaching a point where these measurements become stable under variation of small values of Δ while paying a price in statistics.

The purpose of this Letter is to calculate explicitly the $I = 1$ relative contributions in $B \rightarrow \rho\rho$ decay rates, studying in detail their dependence on the widths *and location* of the two ρ meson bands. It will be shown that these contributions for a common band behave like $(\Delta/m_\rho)^2$ also for $\Delta > \Gamma_\rho$, with a calculable coefficient ~ 0.1 increasing moderately with decreasing Δ , reaching an asymptotic behavior $\frac{1}{6}(\Delta/m_\rho)^2$ for $\Delta \ll 2\Gamma_\rho$. As their absolute normalization is a priori unknown one must also consider the possibility that they are dynamically enhanced relative to the $I = 0, 2$ terms, in which case they would remain significant also for $\Delta = \Gamma_\rho$. A judicious choice of two different locations of the two ρ meson bands will be used to artificially enhance this $I = 1$ contribution by a large factor relative to $(\Gamma_\rho/m_\rho)^2$. This enables resolving completely the uncertainty in α also when the $I = 1$ amplitude is dynamically enhanced.

In Section 2 we study the contribution of an $I = 1$ amplitude to $B \rightarrow \rho\rho$ decay rates for two ρ mesons in a common mass band centered at m_ρ , assuming for each ρ meson decay into two pions a Breit-Wigner distribution. Similar calculations are performed in Sections 3 for two ρ meson bands adjacent to each other and in Section 4 for two bands separated from each other, lying in both cases above and below m_ρ . We summarize our results numerically and conclude in Section 5.

2 Two ρ mesons in a common band centered at m_ρ

The following discussion applies separately to the decays $B^0 \rightarrow \rho^+\rho^-$ and $B^+ \rightarrow \rho^+\rho^0$, in which the final ρ mesons have been measured to be almost 100% longitudinally polarized [2]. It does not apply to $B^0 \rightarrow \rho^0\rho^0$ in which the final state cannot be in $I = 1$. We will consider decay amplitudes $A_L(B \rightarrow \rho\rho)$ for longitudinally polarized ρ mesons in the first two processes. These amplitudes depend on two variables, the two dipion invariant masses, $m_{12}^2 \equiv (p_1 + p_2)^2$ and $m_{34}^2 \equiv (p_3 + p_4)^2$, through Breit-Wigner distributions:

$$A_L(B \rightarrow \rho\rho) = f(m_{12}, m_{34}) \left(\frac{m_\rho \Gamma_\rho}{m_{12}^2 - m_\rho^2 + im_\rho \Gamma_\rho} \right) \left(\frac{m_\rho \Gamma_\rho}{m_{34}^2 - m_\rho^2 + im_\rho \Gamma_\rho} \right). \quad (1)$$

The decay amplitude involves two parts corresponding to final states involving isospin zero or two (two alone for $B^+ \rightarrow \rho^+\rho^0$) and isospin one, respectively:

$$f(m_{12}, m_{34}) = f_{I=0,2}(m_{12}, m_{34}) + f_{I=1}(m_{12}, m_{34}). \quad (2)$$

These two parts are symmetric and antisymmetric, respectively, under interchanging m_{12} and m_{34} ,

$$f_{I=0,2}(m_{12}, m_{34}) = f_{I=0,2}(m_{34}, m_{12}), \quad f_{I=1}(m_{12}, m_{34}) = -f_{I=1}(m_{34}, m_{12}), \quad (3)$$

implying

$$f(m_{12}, m_{34}) = f_{I=0,2}(m_{12}, m_{34}) - f_{I=1}(m_{34}, m_{12}). \quad (4)$$

We will consider the longitudinal decay rate $\Gamma_L(B \rightarrow \rho\rho)$ for invariant masses m_{12}, m_{34} lying in a common range M symmetric about m_ρ :

$$\begin{aligned}\Gamma_L(B^0 \rightarrow \rho^+\rho^-)_M &= (m_\rho\Gamma_\rho)^4 \int_M \int_M \frac{|f(m_{12}, m_{34})|^2 dm_{12} dm_{34}}{|m_{12}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2 |m_{34}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2} \\ &= (m_\rho\Gamma_\rho)^4 \int_M \int_M \frac{(|f_{I=0,2}(m_{12}, m_{34})|^2 + |f_{I=1}(m_{12}, m_{34})|^2) dm_{12} dm_{34}}{|m_{12}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2 |m_{34}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2}.\end{aligned}\quad (5)$$

The interference term is antisymmetric in $m_{12} \leftrightarrow m_{34}$ and vanishes when these two variables are integrated over a common range [5],

$$\int_M \int_M \frac{\text{Re}[f_{I=0,2}(m_{12}, m_{34}) f_{I=1}^*(m_{12}, m_{34})] dm_{12} dm_{34}}{|m_{12}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2 |m_{34}^2 - m_\rho^2 + im_\rho\Gamma_\rho|^2} = 0. \quad (6)$$

The leading term in the $I = 0, 2$ amplitude behaves like a constant, $f_{I=0,2} = a$, while that of the $I = 1$ amplitude behaves like $f_{I=1} = c(m_{12} - m_{34})/m_\rho$. The authors of Ref. [5] assume that the two constants a and c are of the same order. This seems like a reasonable assumption which needs to be tested experimentally. Thus the ratio R_L of the contributions of these amplitudes to the decay rate is:

$$\begin{aligned}R_L &= \frac{c^2}{a^2} \int_M \int_M \frac{[(m_{12} - m_{34})/m_\rho]^2 dm_{12} dm_{34}}{[(m_{12}^2 - m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2][m_{12} \rightarrow m_{34}]} / \left(\int_M \frac{dm}{(m^2 - m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2} \right)^2 \\ &= \frac{c^2}{a^2} \frac{I_1}{(I_0)^2}.\end{aligned}\quad (7)$$

I_1 and $(I_0)^2$ denote the integrals in the numerator and denominator, characterizing (up to a ratio c^2/a^2) contributions of $I = 1$ and $I = 0, 2$ amplitudes to $B \rightarrow \rho\rho$ decay rates. In this section and in the next two sections we will study their ratio for varying ranges of the two ρ meson bands.

Defining $\gamma \equiv \Gamma_\rho/m_\rho$, $x \equiv m/m_\rho$, $x_1 \equiv m_{12}/m_\rho$, $x_2 \equiv m_{34}/m_\rho$, $\delta \equiv \Delta/2m_\rho$, and considering a range $1 - \delta \leq x, x_1, x_2 \leq 1 + \delta$ for these three variables corresponding to a common mass band of width Δ for m_{12} and m_{34} , $m_\rho(1 - \delta) \leq m_{12}, m_{34} \leq m_\rho(1 + \delta)$, we have

$$I_0 \equiv \int_{1-\delta}^{1+\delta} \frac{dx}{(x^2 - 1)^2 + \gamma^2}, \quad (8)$$

$$\begin{aligned}I_1 &\equiv \int_{1-\delta}^{1+\delta} \int_{1-\delta}^{1+\delta} \frac{(x_1 - x_2)^2 dx_1 dx_2}{[(x_1^2 - 1)^2 + \gamma^2][(x_2^2 - 1)^2 + \gamma^2]} \\ &= 2 \int_{1-\delta}^{1+\delta} \frac{x^2 dx}{(x^2 - 1)^2 + \gamma^2} \int_{1-\delta}^{1+\delta} \frac{dx}{(x^2 - 1)^2 + \gamma^2} - 2 \left(\int_{1-\delta}^{1+\delta} \frac{x dx}{(x^2 - 1)^2 + \gamma^2} \right)^2.\end{aligned}\quad (9)$$

Thus we are interested in the following integrals,

$$J_0 \equiv \int_{1-\delta}^{1+\delta} \frac{dx}{(x^2 - 1)^2 + \gamma^2}, \quad J_1 \equiv \int_{1-\delta}^{1+\delta} \frac{x dx}{(x^2 - 1)^2 + \gamma^2}, \quad J_2 \equiv \int_{1-\delta}^{1+\delta} \frac{x^2 dx}{(x^2 - 1)^2 + \gamma^2}, \quad (10)$$

where we want to calculate $I_0 = J_0$, $I_1 = 2[J_2 J_0 - (J_1)^2]$, and $I_1/(I_0)^2$. We will study cases where γ and δ (of order γ or smaller) are small in comparison with 1, so the main contributions to the integrals come from values of x close to 1.

We substitute $u = x^2 - 1$, so $du = 2x dx = 2\sqrt{1+u} dx$. For small u the limits of integration $x = 1 \pm \delta$ translate to $u = \pm 2\delta$, and the integrals can be written

$$I_0 = \frac{1}{2} \int_{-2\delta}^{2\delta} \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2}, \quad (11)$$

$$I_1 = \frac{1}{2} \int_{-2\delta}^{2\delta} \frac{(1+u)^{1/2} du}{u^2 + \gamma^2} \int_{-2\delta}^{2\delta} \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2} - \frac{1}{2} \left(\int_{-2\delta}^{2\delta} \frac{du}{u^2 + \gamma^2} \right)^2. \quad (12)$$

Applying two simple integral functions,

$$\int^u \frac{du}{u^2 + \gamma^2} = \frac{1}{\gamma} \arctan\left(\frac{u}{\gamma}\right), \quad \int^u \frac{u^2 du}{u^2 + \gamma^2} = \int^u \frac{(u^2 + \gamma^2 - \gamma^2) du}{u^2 + \gamma^2} = u - \gamma \arctan\left(\frac{u}{\gamma}\right), \quad (13)$$

implies

$$\int_{-2\delta}^{2\delta} \frac{du}{u^2 + \gamma^2} = \frac{2}{\gamma} \arctan\left(\frac{2\delta}{\gamma}\right), \quad \int_{-2\delta}^{2\delta} \frac{u^2 du}{u^2 + \gamma^2} = 4\delta - 2\gamma \arctan\left(\frac{2\delta}{\gamma}\right). \quad (14)$$

Using the binomial expansions $(1+u)^{1/2} = 1 + (u/2) - (u^2/8) + \dots$ and $(1+u)^{-1/2} = 1 - (u/2) + 3(u^2/8) - \dots$, collecting terms, omitting integrals whose integrands are odd in u , and canceling some terms, we find

$$I_0 = \frac{1}{\gamma} \arctan\left(\frac{2\delta}{\gamma}\right) + \frac{3}{8} \left[2\delta - \gamma \arctan\left(\frac{2\delta}{\gamma}\right) \right] = \frac{1}{\gamma} \arctan\left(\frac{2\delta}{\gamma}\right) [1 + \mathcal{O}(\gamma^2)], \quad (15)$$

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{-2\delta}^{2\delta} \frac{(1 - u^2/8) du}{u^2 + \gamma^2} \int_{-2\delta}^{2\delta} \frac{(1 + 3u^2/8) du}{u^2 + \gamma^2} - \frac{1}{2} \left(\int_{-2\delta}^{2\delta} \frac{du}{u^2 + \gamma^2} \right)^2 \\ &= \frac{1}{8} \int_{-2\delta}^{2\delta} \frac{u^2 du}{u^2 + \gamma^2} \int_{-2\delta}^{2\delta} \frac{du}{u^2 + \gamma^2} = \frac{1}{2} \arctan\left(\frac{2\delta}{\gamma}\right) \left[\left(\frac{2\delta}{\gamma}\right) - \arctan\left(\frac{2\delta}{\gamma}\right) \right] [1 + \mathcal{O}(\gamma^2)]. \end{aligned} \quad (16)$$

The values of I_0 for $\delta = (2\gamma, \gamma, \gamma/2)$ are respectively $1.3258/\gamma, 1.1071/\gamma, \pi/(4\gamma)$, while those of I_1 are $1.7727, 0.4943, 0.08427$. Thus we find

$$\frac{I_1}{(I_0)^2} = \begin{cases} 1.008\gamma^2 = 0.063(\Delta/m_\rho)^2 & \delta = 2\gamma \text{ or } \Delta = 4\Gamma_\rho, \\ 0.403\gamma^2 = 0.101(\Delta/m_\rho)^2 & \delta = \gamma \text{ or } \Delta = 2\Gamma_\rho, \\ 0.137\gamma^2 = 0.137(\Delta/m_\rho)^2 & \delta = \frac{1}{2}\gamma \text{ or } \Delta = \Gamma_\rho. \end{cases} \quad (17)$$

In the limit $\delta \ll \gamma$ (i.e., $\Delta \ll 2\Gamma_\rho$) one may calculate $I_1/(I_0)^2$ using a Taylor expansion $\arctan(\delta/\gamma) = (\delta/\gamma) - (\delta/\gamma)^3/3 + (\delta/\gamma)^5/5 - \dots$. The dominant terms in I_0 and I_1 are

$$I_0 = \frac{2\delta}{\gamma^2}, \quad I_1 = \frac{\gamma}{6} \left(\frac{2\delta}{\gamma}\right)^3 \left(\frac{2\delta}{\gamma^2}\right) = \frac{8}{3} \frac{\delta^4}{\gamma^4}, \quad (18)$$

implying

$$\frac{I_1}{(I_0)^2} = \frac{2}{3} \delta^2 = \frac{1}{6} (\Delta/m_\rho)^2. \quad (19)$$

3 Two adjacent ρ mass bands above and below m_ρ

The longitudinal decay rate $\Gamma_L(B^0 \rightarrow \rho^+ \rho^-)$ is obtained by integrating the amplitude squared over ranges M_1 for m_{12} and M_2 for m_{34} and *vice versa*, for two adjacent ranges M_1 and M_2 above and below m_ρ situated symmetrically with respect to m_ρ . The interference term between $I = 0, 2$ and $I = 1$ amplitudes is antisymmetric in $m_{12} \leftrightarrow m_{34}$ and vanishes when integrating these two variables over the two ranges M_1 and M_2 symmetrically,

$$\left(\int_{M_1} dm_{12} \int_{M_2} dm_{34} + \int_{M_2} dm_{12} \int_{M_1} dm_{34} \right) \frac{\text{Re}[f_{I=0,2}(m_{12}, m_{34}) f_{I=1}^*(m_{12}, m_{34})]}{|m_{12}^2 - m_\rho^2 + im_\rho \Gamma_\rho|^2 |m_{34}^2 - m_\rho^2 + im_\rho \Gamma_\rho|^2} = 0 . \quad (20)$$

Using $f_{I=0,2} = a$, $f_{I=1} = c(m_{12} - m_{34})/m_\rho$, the ratio of their contributions to the integrated decay rate is

$$\begin{aligned} \mathcal{R}_L &= \frac{c^2}{a^2} \int_{M_1} \int_{M_2} \frac{[(m_{12} - m_{34})/m_\rho]^2 dm_{12} dm_{34}}{[(m_{12}^2 - m_\rho^2)^2 + (m_\rho \Gamma_\rho)^2][m_{12} \rightarrow m_{34}]} \bigg/ \int_{M_1} \frac{dm_{12}}{(m_{12}^2 - m_\rho^2)^2 + (m_\rho \Gamma_\rho)^2} \times \left[\frac{M_1 \rightarrow M_2}{m_{12} \rightarrow m_{34}} \right] \\ &\equiv \frac{c^2}{a^2} \frac{\mathcal{I}_1}{\mathcal{I}_{01} \mathcal{I}_{02}} . \end{aligned} \quad (21)$$

The double integral in the numerator denoted by \mathcal{I}_1 involves variables m_{12} and m_{34} which are larger and smaller than m_ρ , respectively. The two single-variable integrals in the denominator corresponding to the ranges M_1 and M_2 are denoted by \mathcal{I}_{01} and \mathcal{I}_{02} , respectively.

We now have

$$\mathcal{I}_{01} \equiv \int_1^{1+\delta} \frac{dx}{(x^2 - 1)^2 + \gamma^2} , \quad \mathcal{I}_{02} \equiv \int_{1-\delta}^1 \frac{dx}{(x^2 - 1)^2 + \gamma^2} . \quad (22)$$

Substituting $u = x^2 - 1$, $du = 2x dx = 2\sqrt{1+u} dx$, expanding $(1+u)^{-1/2} = 1 - u/2 + \dots$ and using the integrals (13) and

$$\int^u \frac{u du}{u^2 + \gamma^2} = \frac{1}{2} \ln(u^2 + \gamma^2) , \quad (23)$$

we obtain

$$\begin{aligned} \mathcal{I}_{01} &= \frac{1}{2} \int_0^{2\delta} \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2} = \frac{1}{2} \int_0^{2\delta} \frac{du}{u^2 + \gamma^2} - \frac{1}{4} \int_0^{2\delta} \frac{u du}{u^2 + \gamma^2} \\ &= \frac{1}{2\gamma} \arctan\left(\frac{2\delta}{\gamma}\right) - \frac{1}{8} \ln\left(\frac{4\delta^2 + \gamma^2}{\gamma^2}\right) . \end{aligned} \quad (24)$$

Similarly

$$\mathcal{I}_{02} = \frac{1}{2} \int_{-2\delta}^0 \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2} = \frac{1}{2\gamma} \arctan\left(\frac{2\delta}{\gamma}\right) + \frac{1}{8} \ln\left(\frac{4\delta^2 + \gamma^2}{\gamma^2}\right) . \quad (25)$$

That is, the leading terms in \mathcal{I}_{01} and \mathcal{I}_{02} behaving like $1/\gamma$ are equal to each other. The subleading term occurring with opposite signs affects the product $\mathcal{I}_{01} \mathcal{I}_{02}$ merely by its square. We will neglect this correction of order γ^2 as we have done in Sec. 2:

$$\mathcal{I}_{01} \mathcal{I}_{02} = \left[\frac{1}{2\gamma} \arctan\left(\frac{2\delta}{\gamma}\right) \right]^2 [1 + \mathcal{O}(\gamma^2)] . \quad (26)$$

We now calculate \mathcal{I}_1 using binomial expansions as in Sec. 2:

$$\begin{aligned}
\mathcal{I}_1 &\equiv \int_1^{1+\delta} \int_{1-\delta}^1 \frac{(x_1 - x_2)^2 dx_1 dx_2}{[(x_1^2 - 1)^2 + \gamma^2][(x_2^2 - 1)^2 + \gamma^2]} = \int_1^{1+\delta} \frac{x_1^2 dx_1}{(x_1^2 - 1)^2 + \gamma^2} \int_{1-\delta}^1 \frac{dx_2}{(x_2^2 - 1)^2 + \gamma^2} \\
&+ \int_1^{1+\delta} \frac{dx_1}{(x_1^2 - 1)^2 + \gamma^2} \int_{1-\delta}^1 \frac{x_2^2 dx_2}{(x_2^2 - 1)^2 + \gamma^2} - 2 \int_1^{1+\delta} \frac{x_1 dx_1}{(x_1^2 - 1)^2 + \gamma^2} \int_{1-\delta}^1 \frac{x_2 dx_2}{(x_2^2 - 1)^2 + \gamma^2} \\
&\simeq \frac{1}{8} \int_0^{2\delta} \frac{du}{u^2 + \gamma^2} \int_0^{2\delta} \frac{u^2 du}{u^2 + \gamma^2} + \frac{1}{8} \left(\int_0^{2\delta} \frac{u du}{u^2 + \gamma^2} \right)^2 \\
&= \frac{1}{4} \left(\frac{1}{2} \arctan \left(\frac{2\delta}{\gamma} \right) \left[\left(\frac{2\delta}{\gamma} \right) - \arctan \left(\frac{2\delta}{\gamma} \right) \right] + \frac{1}{8} \left[\ln \left(\frac{4\delta^2 + \gamma^2}{\gamma^2} \right) \right]^2 \right) [1 + \mathcal{O}(\gamma^2)]. \quad (27)
\end{aligned}$$

Note that the first term in \mathcal{I}_1 equals $\frac{1}{4}I_1$ calculated in (16) in Sec. 1. By itself it would have implied $\mathcal{I}_1/(\mathcal{I}_{01}\mathcal{I}_{02}) = I_1/(I_0)^2$ because $\mathcal{I}_{01}\mathcal{I}_{02} = \frac{1}{4}I_0$. The additional \ln -squared term leads to a contribution with the same positive sign which is somewhat smaller than the first term for δ of order γ .

The values of $\sqrt{\mathcal{I}_{01}\mathcal{I}_{02}}$ for $\delta = (2\gamma, \gamma, \gamma/2)$ are respectively $0.6629/\gamma, 0.5536/\gamma, \pi/(8\gamma)$, while those of \mathcal{I}_1 are $0.6940, 0.2045, 0.03608$. Thus we find

$$\frac{\mathcal{I}_1}{\mathcal{I}_{01}\mathcal{I}_{02}} = \begin{cases} 1.579\gamma^2 & \delta = 2\gamma, \\ 0.667\gamma^2 & \delta = \gamma, \\ 0.234\gamma^2 & \delta = \frac{1}{2}\gamma. \end{cases} \quad (28)$$

Comparing these results with (17) and the line above we conclude that taking two adjacent ρ bands instead of a single common band leads to suppression by a factor two of the dominant $I = 0, 2$ contribution and to moderate enhancement of $57 - 71\%$ in the relative $I = 1$ contribution.

4 Two separated ρ meson bands each of width Γ_ρ

In order to increase considerably the $I = 1$ contribution to the decay rate relative to the $I = 0, 2$ contribution we choose M_1 and M_2 to be two ρ bands each of width Γ_ρ , separated from each other by mass ranges of width Γ_ρ or $2\Gamma_\rho$. These two cases correspond to the following ranges in x :

$$\text{(a) } 1 - \frac{3}{2}\gamma \leq x \leq 1 - \frac{1}{2}\gamma, \quad 1 + \frac{1}{2}\gamma \leq x \leq 1 + \frac{3}{2}\gamma, \quad (29)$$

$$\text{(b) } 1 - 2\gamma \leq x \leq 1 - \gamma, \quad 1 + \gamma \leq x \leq 1 + 2\gamma. \quad (30)$$

We study these two cases separately.

(a) Using notation and calculations as in Sec. 2 we then obtain

$$\mathcal{I}_{01} \equiv \int_{1+\gamma/2}^{1+3\gamma/2} \frac{dx}{(x^2 - 1)^2 + \gamma^2} = \frac{1}{2} \int_\gamma^{3\gamma} \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2} = \frac{1}{2\gamma} [\arctan(3) - \arctan(1)] - \frac{1}{8} \ln(5), \quad (31)$$

$$\begin{aligned}
\mathcal{I}_1 &\simeq \frac{1}{8} \int_{\gamma}^{3\gamma} \frac{du}{u^2 + \gamma^2} \int_{\gamma}^{3\gamma} \frac{u^2 du}{u^2 + \gamma^2} + \frac{1}{8} \left(\int_{\gamma}^{3\gamma} \frac{u du}{u^2 + \gamma^2} \right)^2 \\
&= \frac{1}{8} [\arctan(3) - \arctan(1)] [2 - \arctan(3) + \arctan(1)] + \frac{1}{32} [\ln(5)]^2, \quad (32)
\end{aligned}$$

implying

$$\sqrt{\mathcal{I}_{01}\mathcal{I}_{02}} = \frac{0.2318}{\gamma}, \quad \mathcal{I}_1 = 0.1700, \quad \frac{\mathcal{I}_1}{\mathcal{I}_{01}\mathcal{I}_{02}} = 3.16\gamma^2. \quad (33)$$

(b) For this range we calculate

$$\mathcal{I}_{01} \equiv \int_{1+\gamma}^{1+2\gamma} \frac{dx}{(x^2 - 1)^2 + \gamma^2} = \frac{1}{2} \int_{2\gamma}^{4\gamma} \frac{(1+u)^{-1/2} du}{u^2 + \gamma^2} = \frac{1}{2\gamma} [\arctan(4) - \arctan(2)] - \frac{1}{8} \ln(17/5), \quad (34)$$

$$\begin{aligned}
\mathcal{I}_1 &\simeq \frac{1}{8} \int_{2\gamma}^{4\gamma} \frac{du}{u^2 + \gamma^2} \int_{2\gamma}^{4\gamma} \frac{u^2 du}{u^2 + \gamma^2} + \frac{1}{8} \left(\int_{2\gamma}^{4\gamma} \frac{u du}{u^2 + \gamma^2} \right)^2 \\
&= \frac{1}{8} [\arctan(4) - \arctan(2)] [2 - \arctan(4) + \arctan(2)] + \frac{1}{32} [\ln(17/5)]^2, \quad (35)
\end{aligned}$$

implying

$$\sqrt{\mathcal{I}_{01}\mathcal{I}_{02}} = \frac{0.1093}{\gamma}, \quad \mathcal{I}_1 = 0.0955, \quad \frac{\mathcal{I}_1}{\mathcal{I}_{01}\mathcal{I}_{02}} = 7.99\gamma^2. \quad (36)$$

The values of $\mathcal{I}_1/\mathcal{I}_{01}\mathcal{I}_{02}$ in (33) and (36) should be compared with the much smaller value, $I_1/(I_0)^2 = 0.137\gamma^2$, obtained for this ratio for a common central ρ mass band of width Γ_ρ . The separation of the two ρ bands by gaps Γ_ρ and $2\Gamma_\rho$ enhances the relative $I = 1$ contribution by factors of 23 and 58, respectively. These large enhancements are partially due to a suppression of the $I = 0, 2$ contribution by factors of 11.5 and 52, respectively.

5 Summary and conclusions

Let us compare overall decay rates and relative $I = 1$ contributions to decay rates for ρ meson bands of decreasing width, considering first common ρ bands and then two separated bands for the two pairs of pions. We will refer specifically to relevant measurements by the Babar and Belle Collaborations, using for the ρ mass and width the values [6] $m_\rho = 775$ MeV, $\Gamma_\rho = 148.5$ MeV implying $\gamma^2 = 0.0367$.

The Babar [7] and Belle [8] collaborations studied longitudinally polarized $B^0 \rightarrow \rho^+ \rho^-$ for the two pion pairs forming a common ρ band roughly of width $4\Gamma_\rho$,

$$m_\rho - 2\Gamma_\rho = 478 \text{ MeV} \leq m(\pi\pi) \leq 1072 \text{ MeV} = m_\rho + 2\Gamma_\rho. \quad (37)$$

BaBar used a similar ρ band for studying $B^+ \rightarrow \rho^+ \rho^0$ [9]. The averaged relative errors in the two measured decay rates are around 7–8% [2]. The $I = 1$ contribution to the decay rate is characterized for this band by a quantity which is about half this error,

$$\left[\frac{I_1}{(I_0)^2} \right]_{4\Gamma_\rho} = (1.008)(0.0367) = 0.037. \quad (38)$$

The Belle collaboration has measured $B^+ \rightarrow \rho^+ \rho^0$ [10] using a narrower band approximately of width $2\Gamma_\rho$,

$$m_\rho - \Gamma_\rho = 626 \text{ MeV} \leq m(\pi\pi) \leq 924 \text{ MeV} = m_\rho + \Gamma_\rho . \quad (39)$$

The measured decay rate involved a rather larger error (around 25%) because the Belle analysis was based on only about ten percent of the final Belle $\Upsilon(4S)$ sample. Using our result

$$\frac{[(I_0)^2]_{2\Gamma_\rho}}{[(I_0)^2]_{4\Gamma_\rho}} = 0.70 , \quad (40)$$

one expects with the complete Belle data sample an error in the decay rate around 10%, somewhat larger than measured for a band of width $4\Gamma_\rho$. The relative $I = 1$ contribution to the decay rate expected for a band of width $2\Gamma_\rho$, characterized by

$$\left[\frac{I_1}{(I_0)^2} \right]_{2\Gamma_\rho} = (0.403)(0.0367) = 0.015 , \quad (41)$$

is about 40% $\simeq 0.015/0.037$ of the one for a band of width $4\Gamma_\rho$. In order to reach this sensitivity in measurements of $B \rightarrow \rho\rho$ decay rates one needs about 6 $\simeq (0.037/0.015)^2$ times more data than accumulated so far.

For an even narrower ρ band of width Γ_ρ , as studied briefly by Belle [8],

$$m_\rho - \frac{1}{2}\Gamma_\rho = 701 \text{ MeV} \leq m(\pi\pi) \leq 849 \text{ MeV} = m_\rho + \frac{1}{2}\Gamma_\rho , \quad (42)$$

we calculate

$$\frac{[(I_0)^2]_{\Gamma_\rho}}{[(I_0)^2]_{4\Gamma_\rho}} = 0.35 . \quad (43)$$

Thus for this range one requires about three times more data than used by Babar and Belle for measuring an error of 8% on $B \rightarrow \rho\rho$ decay rates, and about fifty times more data for reaching an accuracy of two percent in these rates (or one percent in corresponding amplitudes). [That is 1/4 of the present 8%, requiring 16 times more data, and multiplying 16 by the factor of three mentioned just below Eq. (43)]. The $I = 1$ contribution for such a narrow band is characterized by an even smaller number:

$$\left[\frac{I_1}{(I_0)^2} \right]_{\Gamma_\rho} = 0.005 . \quad (44)$$

Therefore, unless c^2/a^2 is considerably larger than one, measuring $B \rightarrow \rho\rho$ decay rates for a ρ meson band of width Γ_ρ with fifty times more data than used by Babar and Belle is expected to yield values for $B \rightarrow \rho\rho$ amplitudes which are insensitive to the ρ width. Such a data sample is expected at the SuperKEKB Belle II experiment [3, 4].

If $c^2/a^2 \gg 1$ the $I = 1$ contribution to the decay rate for a ρ band of width Γ_ρ is considerably larger than one percent. In this case one would hope to be able to subtract this contribution from the measured decay rate in order to obtain the pure $I = 0, 2$ contribution.

This would require a higher sensitivity to the $I = 1$ contribution. For this purpose we have studied two pairs of pions for two separated ranges of dipion masses.

For two ρ bands each of width Γ_ρ separated by a range of width Γ_ρ , described by Eqs. (29) and (33),

$$m_\rho - \frac{3}{2}\Gamma_\rho = 552 \text{ MeV} \leq m(\pi\pi) \leq 701 \text{ MeV} = m_\rho - \frac{1}{2}\Gamma_\rho, \quad (45)$$

$$m_\rho + \frac{1}{2}\Gamma_\rho = 849 \text{ MeV} \leq m(\pi\pi) \leq 998 \text{ MeV} = m_\rho + \frac{3}{2}\Gamma_\rho, \quad (46)$$

we calculated using the above value of γ^2

$$\frac{\mathcal{I}_1}{\mathcal{I}_{01}\mathcal{I}_{02}} = (3.16)(0.0367) = 0.116, \quad (47)$$

while for two bands of width Γ_ρ separated by a mass range $2\Gamma_\rho$, described by Eqs. (30) and (36),

$$m_\rho - 2\Gamma_\rho = 478 \text{ MeV} \leq m(\pi\pi) \leq 626 \text{ MeV} = m_\rho - \Gamma_\rho, \quad (48)$$

$$m_\rho + \Gamma_\rho = 924 \text{ MeV} \leq m(\pi\pi) \leq 1072 \text{ MeV} = m_\rho + 2\Gamma_\rho, \quad (49)$$

one finds

$$\frac{\mathcal{I}_1}{\mathcal{I}_{01}\mathcal{I}_{02}} = (7.99)(0.0367) = 0.293. \quad (50)$$

These values are much larger than (44) calculated for a single ρ band of width Γ_ρ , which indicates a higher sensitivity to the $I = 1$ amplitude. The effect of such an enhanced $I = 1$ amplitude on the $B \rightarrow \rho\rho$ rate would be apparent when comparing the branching fraction obtained from a single band of width Γ_ρ with that obtained from two bands of width Γ_ρ separated by a gap of Γ_ρ . We have also shown large suppressions of the $I = 0, 2$ contributions for two bands separated by Γ_ρ and $2\Gamma_\rho$,

$$\frac{[\mathcal{I}_{01}\mathcal{I}_{02}]_{\text{separated by } (\Gamma_\rho, 2\Gamma_\rho)}}{[(I_0)^2]_{\Gamma_\rho}} = \left(\frac{1}{11.5}, \frac{1}{52} \right). \quad (51)$$

Consider the first case (a) “Separated by Γ_ρ ” in the above equation. $\mathcal{I}_{01}\mathcal{I}_{02}/(I_0)^2 = 1/11.5$ implies that this range requires about $33 = 11.5/0.35$ times more data than used by Babar and Belle (using a range $4\Gamma_\rho$) for measuring an error of 8% in the $B \rightarrow \rho\rho$ decay rate. Using 50 times more data, as expected at SuperKEKB, would reduce this error in decay rate to $8\%\sqrt{33/50} = 6.5\%$.

Now assume, for instance, $c^2/a^2 = 10$ in which case (47) implies $(c^2/a^2)\mathcal{I}_1/(\mathcal{I}_{01}\mathcal{I}_{02}) = 1.16 = 116\%$ which is 18 times the above error of 6.5%. Actually, in this case the decay rate has two comparable contributions from $a^2\mathcal{I}_{01}\mathcal{I}_{02}$ and $c^2\mathcal{I}_1$ and the error is smaller. Specifically, with $c^2/a^2 = 10$ the decay rate for case (a) is $a^2[\mathcal{I}_{01}\mathcal{I}_{02} + 10\mathcal{I}_1] = 2.16a^2\mathcal{I}_{10}\mathcal{I}_{20}$ which is $(2.16/11.5)[a^2(I_0)^2]_{\Gamma_\rho} = [a^2(I_0)^2]_{\Gamma_\rho}/5.3$. This then implies that this separated range requires $15 = 5.3/0.35$ times more data than used by Babar and Belle for an 8% error in decay rate. Using 50 times more data than used by Babar and Belle would reduce this error in the measured decay rate to $8\%/\sqrt{50/15} = 8\%/1.8 = 4.3\%$, and a corresponding error of this order in the relatively large $I = 1$ contribution. Subtraction of the thus-determined $I = 1$

contribution, $(c^2/a^2)[I_1/(I_0)^2]_{\Gamma_\rho} = 5\%$ [cf. Eq. (44)], from the decay rate measured for a single band of width Γ_ρ would then yield a very high-precision value for the pure $I = 0, 2$ contribution in this decay, involving an error around $4.3\% \times 5\% = 0.2\%$ from uncertainty in the $I = 1$ amplitude.

To conclude, unless $c^2/a^2 \gg 1$, the best way to place limits on the $I = 1$ amplitude seems to be the method of Sec. 2, i.e., by considering the two ρ mesons in a common band of width $\Delta \equiv 2m_\rho\delta$ centered at m_ρ , decreasing δ until limited by statistics. For the expected ratio $c^2/a^2 = \mathcal{O}(1)$, the extracted $B \rightarrow \rho\rho$ branching ratios should approach a constant value, with negligible $I = 1$ contribution, as δ is decreased sufficiently down to $\Gamma_\rho/2m_\rho$ and below, namely when $\Delta \leq \Gamma_\rho$. In contrast, if for some unknown reason $c^2/a^2 \gg 1$ then the branching ratio would decrease with decreasing values of δ , as a nonnegligible positive $I = 1$ contribution to the decay rate became smaller. In this case the effect of the $I = 1$ amplitude would show up as an artificial enhancement of the $B \rightarrow \rho\rho$ decay rate measured for two ρ bands of width Γ_ρ separated by a gap Γ_ρ when compared with the decay rate obtained from a single band of width Γ_ρ . Translating the enhancement in the former decay to the $I = 1$ contribution in the latter would yield a very precise value for the $I = 0, 2$ amplitude used in the isospin analysis.

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